18th AMSE APPLICATIONS OF MATHEMATICS AND STATISTICS IN ECONOMICS
International Scientific Conference | Jindřichův Hradec, 2-6 September 2015

REFLECTION OF STYLIZED FACTS IN MEASURING DEPENDENCIES BETWEEN FINANCIAL ASSETS RETURNS

MARTIN BOĎA, MÁRIA KANDEROVÁ
Matej Bel University in Banská Bystrica, Faculty of Economics,
Department of Quantitative Methods and Information Systems,
Tajovského 10, 975 90 Banská Bystrica
email: martin.boda@umb.sk, mária.kanderova@umb.sk

Abstract
In practical applications and a great deal of empirical research, it is a custom to use Pearson correlation coefficients in capturing relationships between returns of financial assets. There are several deficiencies of this approach. First, this is appropriate for elliptical distributions only and it fails to recognize the well-known stylized facts of financial assets returns. Second, Pearson correlations are usable in representing non-linear dependencies even if these are present between returns, in which effect they cannot constitute a general measure of dependence. The goal of the paper is to highlight a possibility of applying an alternative approach to modelling dependencies between financial assets returns based on the concept of copulas and the concept of rank correlation. The paper encompasses both a brief theoretical exposition of the topic, and a practical case study with an evaluation of the presented issues.

Key words: *dependence measure, copula, asset returns, stylized facts.*

1. Introduction

In the financial community, it is understood that asset prices do not evolve as prescribed by the efficient market hypothesis (regardless of whether formulated in the setup of random walk or interpreted under the Martingale model). Instead, it is nowadays believed that the processes by which they are governed are more complex and accepted that they are projected into statistically troublesome features of asset returns. Although there are many statistically coherent variants of the random walk hypothesis, their common integrating ingredient is a statement that price changes are in some way random and prices wander (or, using the terminology of the hypothesis, walk) in an entirely unpredictable way. Yet, attempts to model and predict prices, or returns derived from them, flourish and will never cease to exist. There are solid grounds for these efforts as this sort of modelling is needful for the purpose of portfolio construction, risk measurement and even naïve predictions of the future behaviour of financial portfolios. Insomuch as asset prices follow non-stationary processes (being undoubtedly stochastically integrated), it is rather their returns that are subjected to analysis and modelled. Usually this modelling of asset returns is undertaken in a multivariate context and financial modellers are confronted with the task of measuring dependencies between asset returns. In the accomplishment of this task financial modellers resort to using Pearson correlations, in which they rely on simplicity of this dependence measure and go under the belief that linear measures of dependence suffice to capture the inherent dependence structure that is present between returns. This is erroneous when empirical features of asset returns are taken under advisement that were observed already in the 1960s and keep on being reaffirmed and strengthened by further research and their analysis. These empirical features, representing the stylized facts of financial asset returns, point to the observation that there is a dependence structure present amongst asset return series; nonetheless, it is not linear and must be measured in a different way than it is allowed by traditional Pearson correlations. For the purpose of measuring dependencies, the concept of copulas, amongst others, has become popular in empirical finance, which allows conveniently to accommodate these stylized facts into modelling, extracting the dependence structure from the joint distribution of market returns. Copulas permit building the multivariate joint distribution (of market returns) from individual marginal distributions and measuring dependence between these marginal distributions using for example the concept of concordance dependence. Concordance dependence is more general than linear dependence and exploits commonly the rank correlation coefficients of Kendall and Spearman, which are good surrogates of the Pearson correlation coefficient owing to both their generality and background theoretical justification (see e.g. Embrechts et al., 2002, pp. 176 – 223). Despite a generic nature and attractiveness of copulas, usage of copulas appears probably more frequent in academic circles and financial theory rather than with actual practitioners who tend to favour simpler and less realistic (or absolutely unrealistic) models. The paper promotes copula models in capturing and measuring dependencies between asset returns, by enforcing its goal, which is to highlight a possibility of applying an alternative approach to modelling dependencies between financial assets returns based on the concept of copulas and the concept of rank and concordance correlation. Owing to its introductory nature, the paper does not explore all details of copula applicability in finance and in some respect only touches the issue at hand.

The paper is organized into 5 sections. This introductory section is followed by two factual sections, the first of which summarizes the most important stylized facts of market returns and the second one exposits succinctly theory of copulas to the extent that is motivated by the former section and vital to the line followed in the paper. The fourth section presents an application of the exposited issues and is appended finally by the fifth section that fulfils the function of a conclusion. The application section gives an empirical demonstration as to how a copula model might be fitted and selected in order to capture dependencies between asset returns and then as to how such a model may be used in risk measuring.

2. Stylized facts of financial asset returns

Intensive research focusing up on financial markets since the 1960s has provided a mass of evidence on typical features exhibited by asset returns. These repeating patterns were formulated as a collection of stylized facts (see e.g. Pfaff, 2013, pp. 26, 29; McNeil et al., 2005, pp. 117 – 125; Cont, 2001). Several lessons may be learned from the list of stylized facts. A notable implication is that the multivariate Gaussian distribution is not adequate for modelling and capturing asset returns, and there still outstands the question whether typical elliptical distributions are capable of describing the data generating process of market returns (see e.g. Hose and Huschens, 2013; pp. 99 – 140; Buc and Klieštik, 2013, pp. 215 – 224).. This doubt prolongs to use of Pearson correlations for measuring dependence between asset returns and for capturing their dependence structure. Pearson correlations quantify only the strength of a linear relationship between random variates and are valid only in the case of multivariate elliptically distributed random variates. Elliptical distributions have horizontal cuts through the probability mass of elliptical shape and – by definition, yet loosely speaking – they are fully described by their location and scale, which suggests for a multivariate case that relationships between random variates are sufficiently captured by Pearson correlations and that these relationships are linear. In the light of the stylized facts of asset returns, this belief is not justified and some other dependence measure must be made use of, correcting thus for the shortcomings of the Pearson correlation coefficient. One of them is that this measure is not invariant to non-linear transformation of random variates or may be uninformative in the presence of non-linear dependence between random variates. It may be shown that the value of the Pearson correlation coefficient depends on the marginal distributions and that not all values in the range of [-1,1] cannot be guaranteed to be attainable (see McNeil et al., 2005, pp. 203 – 205). This might be remedied by employment of the rank correlation coefficients of Kendall and Spearman in the framework of copulas. The way how this is done is exposited in the next section.

3. Copulas and measuring dependence between financial asset returns

It seems that copulas are well established in the area of finance, at least in the theory of financial modelling e.g. (Genest et al., 2009a). Copulas are instrumental in isolating two major and underlying components from the joint distribution of random vectors, viz. the marginal behaviour of individual random variates and a characterization of their dependence structure. Applied in the area of financial investment, the concept of copulas replaces (and in practical situations this replacement is unique) the joint distribution function of asset returns by a copula function linking marginal distribution functions with the aid of a set of (possibly dependence) parameters. In this substitution, the copula itself acts as a distribution function on a unit hypercube. More precisely, a d-dimensional copula C is a mapping of the unit hypercube into the unit interval, i.e. $C:[0,1]^d \rightarrow [0,1]$. In addition, three properties are placed on copula C and they are as follows:

1. $C(u_1,\ldots,u_d)$ is increasing in each component u_i for $i \in \{1,\ldots,d\}$.
2. $C(1,\ldots,1,u_i,1,\ldots,1) = u_i$ for all $i \in \{1,\ldots,d\}$ and for any $u_i \in [0,1]$.
3. For every $\mathbf{a},\mathbf{b} \in [0,1]^d$ such that $\mathbf{a} \le \mathbf{b}$ (wherein "≤" introduces an elementwise comparison) define a d-box $[\mathbf{a},\mathbf{b}]$ in the following way $[\mathbf{a},\mathbf{b}] = [a_1,b_1] \times \ldots \times [a_d,b_d]$ and define for each $\mathbf{c} \in [\mathbf{a},\mathbf{b}]$ the $\text{sgn}(\cdot)$ function as $\text{sgn}(\mathbf{c}) = 1$, if $c_i = a_i$ for an even number of i's, and $\text{sgn}(\mathbf{c}) = -1$, if $c_i = a_i$ for an odd number of i's (with $i \in \{1,\ldots,d\}$). Taking the sum over all vertices $\mathbf{c} \in [\mathbf{a},\mathbf{b}]$, then it must be that $\sum \text{sgn}(\mathbf{c}) C(\mathbf{c}) \ge 0$.

The first two conditions are clear, the first property is required for any multivariate distribution function and the second property keeps the validity of marginal distribution functions that are substituted for individual arguments of C. The third property is called the rectangle inequality and its sensibility is explained e.g. in (McNeil et al. 2005, p. 185).

On the one hand, all multivariate distribution functions contain copulas, and, on the other hand, copulas in conjunction with univariate distribution functions are constructors of multivariate distribution functions. In order to see this, let random variates $X_1,\ldots,X_d$ have distribution functions $F_1,\ldots,F_d$, and denote the joint distribution function of the random vector $(X_1,\ldots,X_d)'$ as F. Sklar's Theorem suggests that there exists a copula $C:[0,1]^d \rightarrow [0,1]$ such that $F(x_1,\ldots,x_d) = C(F_1(x_1),\ldots, F_d(x_d))$ for all $x_1,\ldots,x_d \in [-\infty,\infty]$. Furthermore, this theorem also warrants that if the random variates $X_1,\ldots,X_d$ are continuous (and there is no doubt that asset returns satisfy this indeed), then C is unique. Also a converse statement is true, and if C is a copula and $F_1,\ldots,F_d$ are univariate distribution functions, then the function F defined for all $x_1,\ldots,x_d \in [-\infty,\infty]$ as $F(x_1,\ldots,x_d) = C(F_1(x_1),\ldots,F_d(x_d))$ is a joint distribution function with marginals $F_1,\ldots,F_d$. A direct consequence of the fact that the copula C links individual marginals $F_1,\ldots,F_d$ into the respective joint distribution function F is that the dependence structure is embodied in the copula itself. There are ways to extract it and separate it from F.

Several copulas (or even classes of copulas) have been suggested in the literature. A total of four copulas are presented here in the paper. Whereas the first two copulas, the Gaussian copula and the Student's T copula, belong to the category of distribution-based copulas and are prominent examples of elliptical copulas; the other two copulas, the Gumbel and Clayton copulas, are classified as Archimedean copulas. The advantage of distribution-based copulas is their simplicity and easiness of use, but their major drawback is that a closed form of these copulas cannot be derived. On the contrary, Archimedean copulas do not share this disadvantage, but with higher dimensions their structure is too complex.

A Gaussian (or normal) copula is derived from the multinormal and uninormal distribution functions. Denoting the distribution functions of the d-dimensional and univariate standard normal distribution as $\mathbf{\Phi}$ and Φ, respectively, then a d-variate Gaussian copula is defined for all $u_1,\ldots,u_d \in [0,1]$ by the expression $C(u_1,\ldots,u_d,\mathbf{R}) = \mathbf{\Phi}(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_d))$, where $\mathbf{R}$ is a $d \times d$ correlation matrix capturing the dependence structure between marginals. Use of Gaussian copulas in modelling dependencies between asset returns is criticized on the grounds that Gaussian copulas are symmetric copulas and are not adequate for modelling asymmetric tail dependences in asset returns, which is observed in financial markets and which means that when two asset prices fall by large amounts their dependence is greater than when their prices rise (e.g. Alexander, 2008, p. 268).

A similar implicit method of definition is utilized in the case of a Student's T copula. Let $\mathbf{t}_\nu$ and t_ν represent the d-dimensional multivariate and univariate distribution functions of the Student's T distribution with ν degrees of freedom and let $\mathbf{R}$ denote the $d \times d$ correlation matrix, then a d-variate Student's T copula is defined as $C(u_1,\ldots,u_d,\mathbf{R}) = \mathbf{t}_\nu(\mathrm{t}_\nu^{-1}(u_1),\ldots,\mathrm{t}_\nu^{-1}(u_d))$ for all $u_1,\ldots,u_d \in [0,1]$.

Unlike Archimedean copulas, elliptical copulas such as the Gaussian and Student's T are implicit copulas and they are built from the definitional multivariate distribution. Archimedean copulas are derived by means of a generator function. For a generator function $\Psi : [0,1] \to [0,\infty)$, the corresponding Archimedean copula is defined by means of the expression $C(u_1,\ldots,u_d) = \Psi^{-1}(\Psi(u_1) + \ldots + \Psi(u_d))$ for all $u_1,\ldots,u_d \in [0,1]$. Out of Archimedean copulas, the Clayton and Gumbel copula are most frequently applied in market risk analysis and their usefulness comes from the fact that they are capable of capturing asymmetric tail dependence.

The Clayton copula builds upon the generator function $\Psi(u) = (u^{-\delta} - 1)/\delta$ where $\delta > 0$ and $u \in [0,1]$. Such a choice produces an Archimedean copula of the form $C(u_1,\ldots,u_d,\delta) = (1 - d + u_1^{-\delta} + \ldots + u_d^{-\delta})^{-1/\delta}$. The generator function for the Gumbel copula is given as $\Psi(u) = -(\ln u)^{\theta}$ with $\theta \geq 1$ and for $u \in [0,1]$, which generates the form for this copula as $C(u_1,\ldots,u_d,\theta) = \exp\{-[(-\ln u_1)^{\theta} + \ldots + (-\ln u_d)^{\theta})]^{1/\theta}\}$.

As is embedded in, and follows from, the definition of the elliptical distribution, it is correlation that qualifies as a natural measure of dependence between jointly elliptically distributed random variates, but when random variates are not elliptically distributed, correlation fails its function and is subject to a number of limitations as a general measure of dependence. These limitations (and pitfalls to interpretation) are discussed extensively in McNeil et al. (2005, pp. 201 – 205).

An alternative do Pearson correlation is rank correlation that goes in two varieties, the Kendall and Spearman correlation coefficients. An indisputable advantage of rank correlation is that it only depends on the copula that underlies the joint distribution and is not influenced by the marginal distributions. This contrasts with Pearson correlation that depends on both these main statistical ingredients of joint distribution. Kendall and Spearman correlations are both measures of dependence are restricted to the interval [–1,1]. As with Pearson correlation, they yield the value zero for independent random variates but a zero value of rank correlation does not guarantee independence.

Kendall's rank correlation is a measure of concordance for bivariate random vectors and Spearman's rank correlation measures the Pearson correlation between the stochastic ranks of random variables. For the precise definition, consult e.g. (McNeil et al. 2008, pp. 206-208) or (Pfaff 2013, pp. 131-132). It is important to note that if the random variates in question are continuous, then the definitional formulas for both ρ_τ and ρ_S can be reexpressed by means of the underlying copulas, in consequence of which the marginal distributions do not exert their influence upon dependence as captured and measured by ρ_τ and ρ_S. It may be shown e.g. (Alexander, 2008, p. 280) that there is a correspondence between copulas and rank correlations. Keeping the notation as introduced in the earlier exposition, it can be shown that $\rho_\tau = (2/\pi)\arcsin\rho_P$ holds for copulas based on elliptical distributions, and – similarly – that $\rho_S = (6/\pi)\arcsin(\rho_P/2)$ hold for the Gaussian copula. More interestingly, one has for the bivariate Clayton copula that $\delta = 2\rho_\tau/(1-\rho_\tau)$, and for the bivariate Gumbel copula that $\theta = 1/(1-\rho_\tau)$. This is just an acknowledgment that rank correlation dependence measures derive only from the underlying distribution and from the marginals. Furthermore, this also shows that the parameters of Archimedean copulas are linked to the very dependence structure.

A copula-based statistical model consists of three ingredients: (a) a copula, (b) a set of copula parameters inherently tied with the dependence structure, and (c) a set of marginal distributions. As is observed in (McNeil et al. 2005, p. 229), it is often difficult in practical situations to identify a good multivariate model that describes both marginal behaviour and structure effectively. Each of the three ingredients enumerated in (a) to (c) poses a problem, i.e. which copula to use, how to estimate its parameters, and which marginals to choose as most descriptive. These issues are touched in certain detail in (McNeil et al. 2005, pp. 228 – 236), (Alexander, 2008, pp. 281 – 285). As in the empirical section of the paper the canonical maximum likelihood method is employed, only this method is briefly explained. This method of calibrating a copula does not specify the marginals and is similar to the "inference functions for margins" method save the fact that it is implemented in a non-parametric way as far as the marginals are concerned. Observations on individual variates are transformed into observations on uniform variates (which are then called pseudo-observations) using the empirical marginal distributions and then the maximum likelihood estimation of the copula parameters is carried out on these pseudo-observations. This approach has the advantage that the choice of copula is not influenced by the choice of parametric form for the marginals. The best copula may be chosen by comparing optimized likelihoods for various choices as to candidate copulas, using an information criterion in a usual way.

4. Empirical demonstration, its practical aspects and results

Although data of daily frequency are most often used in day-to-day modelling, weekly returns are used in the paper instead since they exhibit higher stability over time and their statistical properties are finer from a modeller's perspective. The in-sample period stretched from the first week of June 2004 until the last week of March 2010 and included 300 effective observations of weekly returns, and the out-of-sample period ran from the first week of April 2010 until the last week of March 2015 and entailed 261 effective returns observed on a weekly basis. Returns were calculated from adjusted closing prices. For the sake of simplicity, only five assets were selected to demonstrate the aforesaid issues. These assets are five shares constituting the S&P 500 Index and falling into the Global Industry Classification Standard category of Information Technology. The list of shares of Information Technology companies comprised: Broadcom Corporation (ticker "BRCM"), Jabil Circuit Inc. (ticker "JBL"), Microchip Technology Inc. (ticker "MCHP"), Salesforce.com Inc. (ticker: "CRM"), Yahoo! Inc. (ticker: "YHOO"). This choice was not intentional and in opting for this specific GICS category and for this quintet of shares no ancillary objective was followed.

In computations and preparing graphical presentations, the software R version 3.0.1 (R Core Team, 2013) was employed with several of its libraries: `copula`, `tseries` and `PerformanceAnalytics`.

The intent of the conducted empirical demonstration is especially to show how a copula model conjoint with a pre-defined dependence structure is instrumental in measuring dependence between asset returns and how this may be used in measuring market risk in the framework of value at risk. In addition, the difference between the three introduced correlation measures is illustrated in this case study by comparing different views they provide on dependence, in which it is emphasized that the choice of a particular correlation measure should go along with the choice of the underlying copula function.

First of all, the full sample of 561 weekly returns (for the period from June 2004 to March 2015) was studied in terms of their dependence structure, which was evaluated through the three dependence measures. As might have been anticipated, the three methods of measuring dependence between asset returns produced divers results, which are displayed for comparison in Table 1. Table 1 juxtaposes full-sample estimates of three different correlation matrices obtained by use of the sample estimators of Pearson correlations, Kendall correlations and Spearman correlations, respectively. There is a striking similarity between Pearson correlations and Spearman rank correlations as the differences, if present, are not weighty from a practical viewpoint. Contrariwise, the matrix of Kendall rank correlations differ from the other two matrices apparently to a larger extent; still pointing to positive dependence between the five investigated returns series, although showing that the individual pairwise correlations coefficients should be lower roughly by 0.10 in magnitude.

Table 1 Pairwise dependence structure as captured by correlation measures using the entire period

Pearson correlations	BRCM	JBL	MCHP	CRM	YHOO
BRCM	1.000	0.434	0.512	0.425	0.316
JBL	0.434	1.000	0.475	0.385	0.348
MCHP	0.512	0.475	1.000	0.433	0.434
CRM	0.425	0.385	0.433	1.000	0.417
YHOO	0.316	0.348	0.434	0.417	1.000

Kendall rank correlations	BRCM	JBL	MCHP	CRM	YHOO
BRCM	1.000	0.346	0.389	0.287	0.207
JBL	0.346	1.000	0.342	0.278	0.224
MCHP	0.389	0.342	1.000	0.300	0.276
CRM	0.287	0.278	0.300	1.000	0.287
YHOO	0.207	0.224	0.276	0.287	1.000

Spearman rank correlations	BRCM	JBL	MCHP	CRM	YHOO
BRCM	1.000	0.490	0.538	0.414	0.297
JBL	0.490	1.000	0.473	0.391	0.324
MCHP	0.538	0.473	1.000	0.429	0.400
CRM	0.414	0.391	0.429	1.000	0.416
YHOO	0.297	0.324	0.400	0.416	1.000

Source: the authors.

In total, four copula models as outlined in the previous section were considered and fitted to the sample of 561 weekly returns using the canonical maximum likelihood method, which means that before initiating the maximum likelihood estimation observed values of returns were translated into the interval [0,1] by computing empirical cumulative frequencies. The results for the Gaussian model, the Student's T model, the Clayton model and the Gumbel model are reported in Table 2. For each copula model, point estimates are provided together with an output for statistical testing their significance in a Fisherian way (which is performed under the reservation of an iid sample). Under the parameter report, also an overview of conventional statistics for assessing the quality of fit is located: the value of optimized log-likelihood (LogLik), the associated values of the Akaike information criterion (AIC) and of the Bayesian information criterion (BIC) as well as the value of goodness-of-fit (GOF) statistic constructed as the functional S_n of Cramér and von Mises (Genest et al., 2009b, p. 201). The estimated „rho“ parameters for the Gaussian and Student's T copulas are canonical maximum likelihood estimates of the non-diagonal elements of the correlation matrix that acts as a parameter of the these two elliptical copulas. One notable feature is that they are quite distant from traditional moment sample estimates reported in Table 1. All the estimated parameters are statistically significant, although both the likelihood-based criteria and the goodness-of-fit statistics indicate that the best fitting copula model is the Student's T. The correlation values estimated with the Student's T copula and reported in Table 2 are preferable to sample estimates reported earlier in Table 1 insomuch as they were obtained as part of the copula fitting procedure and explicitly allow for the dependence structure embedded in the copula describing and substituting the joint distribution of the five share returns.

Table 2 Estimated parameters of the copula models considered using the entire period

Gaussian model					**Student's T model**				
	Estimate	**Standard error**	**Z-value**	**P-value**		**Estimate**	**Standard error**	**Z-value**	**P-value**
rho.1	0.487	0.029	16.878	0.000	**rho.1**	0.519	0.030	16.958	0.000
rho.2	0.543	0.026	20.781	0.000	**rho.2**	0.581	0.027	21.055	0.000
rho.3	0.438	0.031	14.073	0.000	**rho.3**	0.435	0.034	12.588	0.000
rho.4	0.320	0.036	8.990	0.000	**rho.4**	0.339	0.038	8.755	0.000
rho.5	0.487	0.029	16.879	0.000	**rho.5**	0.523	0.031	16.881	0.000
rho.6	0.410	0.032	12.694	0.000	**rho.6**	0.440	0.034	12.749	0.000
rho.7	0.347	0.035	9.954	0.000	**rho.7**	0.361	0.037	9.534	0.000
rho.8	0.447	0.031	14.560	0.000	**rho.8**	0.457	0.033	13.61	0.000
rho.9	0.432	0.032	13.664	0.000	**rho.9**	0.410	0.036	11.338	0.000
rho.10	0.428	0.032	13.462	0.000	**rho.10**	0.436	0.034	12.732	0.000
					df	6.919	0.905	7.639	0.000
LogLik	367.2		**BIC**	-671.1	**LogLik**	413.9		**BIC**	-758.1
AIC	-714.4		**GOF statistic**	0.078	**AIC**	-805.8		**GOF statistic**	0.070
Clayton model					**Gumbel model**				
	Estimate	**Standard error**	**Z-value**	**P-value**		**Estimate**	**Standard error**	**Z-value**	**P-value**
parameter	0.526	0.028	18.780	0.000	**parameter**	1.343	0.020	65.460	0.000
LogLik	286.5		**BIC**	-566.6	**LogLik**	303.4		**BIC**	-600.4
AIC	-571.0		**GOF statistic**	0.674	**AIC**	-604.8		**GOF statistic**	0.553

Source: the authors.

The information on the dependence structure and the estimated Student's T copula model may be used directly in risk quantification. If a portfolio is held made up of the five shares, it is possible to estimate its value at risk in a simple way. First of all, a Monte Carlo simulation from the estimated copula must be undertaken (see McNeil et al., 2015, pp. 193 – 195; Alexander, 2008, pp. 286 – 290) so as to obtain fictional returns that comply to the estimated dependence structure. Insomuch as the "non-parametric" version of the "inference functions for margins" approach is implemented throughout the paper, the result of such a simulation are pseudo-observations, i.e. values of the empirical distribution functions for returns of individual shares, and must be translated into values of returns. This is easily done by calculating empirical quantiles from simulated pseudo-observations separately for each return series under consideration. For each simulation run, individual returns are then aggregated by use of the portfolio weights to a portfolio level and simulated portfolio returns are thus calculated. Eventually, value at risk at a desired confidence level is found as the corresponding empirical quantile of the simulated portfolio returns.

So as to avoid complications, uniform weights were set in the empirical exercise, which means that each of the five assets has a common 20 % share on the portfolio. The Monte Carlo simulation based on 20 000 runs yielded the (predictions of) one-week value at risk 6.43 % at a 0.95 confidence level, 11.39 % at a 0.99 confidence level, and, finally, 17.31 % at a 0.999 confidence level. As all the 561 returns of the full sample were used, these risk estimates apply to the week following the end of the evaluation period, i.e. the first week of April 2015.

A better insight may be gained by repeated fitting a Student's T copula model (and thus recalibrating and reestimating its parameters on a weekly basis) and by repeating value at risk predictions, which was done in the further part of the empirical demonstration. The in-sample period constituted a window of 300 weekly observations covering the period from the first week of June 2004 to the last week of March 2010, which were utilized to calibrate the copula model and to make three value at risk predictions for the first week of April 2010 for the portfolio with constant and uniform weights 20 % per share (these predictions were related to confidence levels 0.95, 0.99 and 0.999). Having effected the first prediction, the window of length 300 slid along the five return series to take on a new quintette of weekly observations and to reject the very first quintette of returns observed for the first week of June 2004. With the new window, the copula model was recalibrated anew and new predictions were made, and this procedure was repeated throughout the entire out-of-sample period 261 times in total. In order to compare the behaviour of these value at risk predictions, the three value at risk series had to be compared with the actual return series, keeping the temporal match between the week in which a particular gain or loss was realized and the week of prediction. This is the reason why the last window could not comprise the last week of March 2014, i.e. the last week of the out-of-sample period, because the formulated value at risk predictions could not be compared. On the surmise that the Student's T model is the best copula model to describe the existing relationship between the five share returns under scrutiny (of course, this is understood in comparative terms with respect to the other three copula models), only this copula model was utilized further.

The results are displayed for convenience in the form of a graphical summary in Figure 1, which comprises two different sorts of information. Whereas the t-plot on the left-hand side compares for the out-of-sample period the time series of actual returns and three value at risk predictions, the t-plot on right-hand side reveals how (canonical maximum likelihood estimates of) dependence measures encompassed in the correlation matrix of the considered copula model changed over the out-of-sample period.

It is clear from the t-plot of returns and value at risk predictions that the rendered predictions are stringent to some extent as one would expect, for the monitored confidence levels 0.95, 0.99 and 0.999, that there are out of 261 predictions 13.50, 2.61 and 0.26 exceedances, respectively. Here, however, the corresponding numbers of exceedances are 5 and 0 (which corresponds to the real coverage 0.981 (for the confidence level 0.95) and 1.000

(for the confidence levels 0.99 and 0.999). Consistent with the multivariate stylized facts, the dependence structure is subject to changes over time.

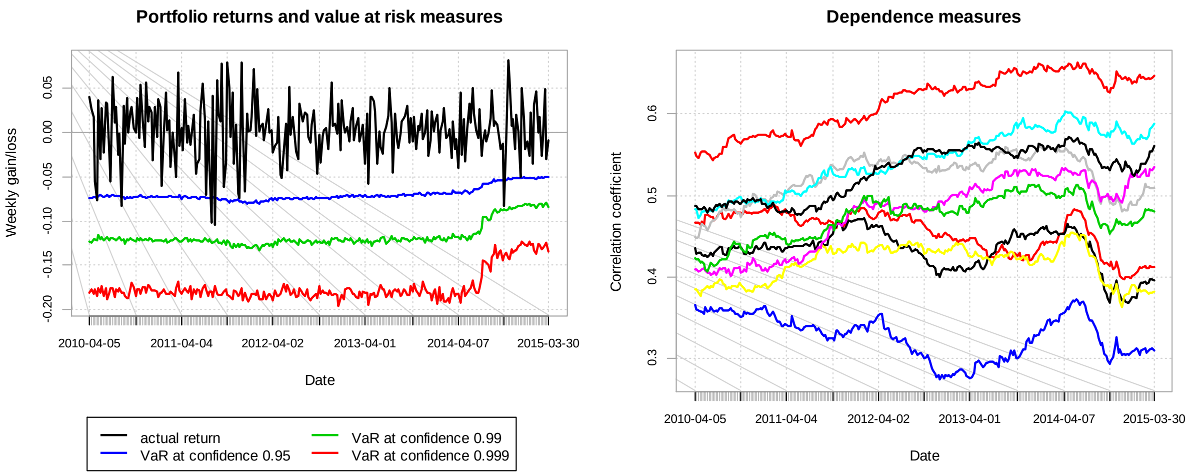

Figure 1 Results of week-to-week dependence measurement and risk measurement in the out-of-sample period
Source: the authors.

Naturally, the following situation of fixed using the Student's T copula model for risk measurement on a sliding weekly basis is somewhat simplified as in practise attempts at modelling marginals would be exerted and possibly more flexible (and sophisticated) copula models would be employed.

5. Conclusion

The legacy sought after and promoted in the paper is that in the practice of financial management more care should be paid to aspects of measuring dependencies between variates of interest. Not only is the traditional Pearson correlation restricted to capturing and assessing linear dependence (or – more correctly – relationship) between the random variates dependence between which is measured, but it is also influenced by these variates themselves (and – more precisely – their marginal distribution). This is clearly undesirable since an ideal measure of dependence should not be affected by the properties possessed by individual random variates, but should only be related to their joint distribution, from which it should extract information on dependence structure. A caution is therefore directed to thoughtless use of Pearson correlation coefficients to measure dependence between arbitrary random variates, especially those which are not jointly elliptically distributed. In point of fact, Pearson correlation is adequate only in these situations when the joint distribution may be satisfactorily depicted and represented by an elliptical copula such as the Gaussian copula or the Student's T copula. Even in such a case, it might be safer not to use conventional sample estimates of correlation coefficients, but it might be wiser to construct these estimates by means of a well-founded method of estimation such as the (canonical) maximum likelihood method which adjust the estimating mechanism to a concrete copula model that should be simultaneously fit to returns data. The concept of rank correlation as proposed by Kendall and Spearman seem to be more general as these coefficients are not influenced by the marginal distributions of individual variables. Notwithstanding this generality, the process of estimating these rank metrics of correlation should not be separated from fitting a general copula model but should be seen as an integrated ingredient of finding the best description for dependence structure of analyzed data.

These factors should be solicitously entertained especially when data modelling is set in the area of financial management and the expected output is an optimized portfolio or estimates of market risk. In such situations, the analysis should not be confined to using traditional Pearsonian correlations but should account for the apparent fact that the data generating process of market returns is somewhat complicated and this also extends to their dependence structure. Therefore, here should care be even more intense and portfolio selection and risk modelling should be based on using a copula model so as to construct more realistic outcomes. The process of fitting (calibrating) a copula model to actual data, though the setting was simple), is also shown in the paper, which then demonstrates how the information on dependence structure may be further utilized in the analysis. The paper has thus communicated a possibility of applying a different outlook on measuring dependence between asset returns than the traditional one intermediated by traditional correlation coefficient, and emphasized the importance of copulas and the associated concept of rank correlation for practical modelling of asset returns in finance.

Acknowledgements

The paper is prepared under the ægis of the grant schemes VEGA No. 1/0757/15 and VEGA No. 1/0647/14.

References

1. ALEXANDER, C. 2008. Market risk analysis: volume II. Practical financial econometrics. Chichester : Wiley, 2008. ISBN 978-0-470-99801-4.
2. BUC, D., KLIEŠTIK, T. 2013. Aspects of statistics in terms of financial modelling and risk. In LÖSTER, T., PAVELKA, T. (eds.) The 7th International Days of Statistics and Economics. Proceedings. ISBN 978-80-86175-87-4, pp. 215-224.
3. CONT, R. 2001. Empirical properties of asset returns: stylized facts and statistical issues. In Quantitative Finance. 2001, vol. 1, iss. 2, pp. 223-236.
4. EMBRECHTS et al. 2002. Correlation and dependence in risk management: properties and pitfalls. In: DEMPSTER, M. A. H. (ed.) Risk management: value at risk and beyond. Cambridge: Cambridge University Press, 2002. ISBN 978-1107001121, pp. 176-223.
5. GENEST, C., GENDRON, M., BOURDEAU-BRIEN, M. 2009a. The Advent of Copulas in Finance. In European Journal of Finance. 2009, vol. 15, iss. 7-8, pp. 609-618.
6. GENEST, C., RÉMILLARD, B., BEAUDOIN, B. 2009b. Goodness-of-fit tests for copulas: A review and a power study. In Insurance: Mathematics and Economics, 2009, vol. 4, iss. 2, pp. 199-214.
7. HOSE, S., HUSCHENS,S. 2013. Stochastic orders and non-Gaussian risk factor models. In Review of Managerial Science, 2013, vol. 7, iss. 2, pp. 99-140.
8. McNEIL, A. J. et al. 2005. Quantitative risk management: concepts, techniques, tools. Princeton : Princeton University Press, 2005. ISBN 978-0-691122557.
9. PFAFF, B. 2013. Financial risk modelling and portfolio optimization with R. Chichester : Wiley, 2013. ISBN 978-0-470-97870-2.
10. R CORE TEAM 2013: R: a language and environment for statistical computing. Vienna: R Foundation for Statistical Computing, 2013, http://www.r-project.org/.